Lec 11: 09/30/2010

Homologous Stellar Models:

Stellar equations require three input functions $P(\rho,T)$, $\kappa(\rho,T)$, $\varepsilon(\rho,T)$ for their integration. In general, these functions have complicated forms. However, one can think of approximating them by simple power laws like:

$$P = P_0\, \rho^a\, T^{-b} \quad , \quad \kappa = \kappa_0\, \rho^n\, T^{-s} \quad , \quad \varepsilon = \varepsilon_0\, \rho^\lambda\, T^\nu$$

in some cases. Such approximations for P and κ will be valid if any single process makes the dominant contribution to pressure and opacity. The approximation of ε by a power law is valid when a given nuclear reaction dominates the energy production.

It is instructive to consider some special cases where the power-law approximations in above are valid:

- $b=0$: In this case we have a polytropic relation $P = \rho_0^{\frac{1}{a}} \rho^{\frac{1}{a}}$. For example, $a = \frac{3}{5}$ corresponds to a fully convective stars, or one consisting of non-relativistic degenerate matter, while $a = \frac{3}{4}$ corresponds to a star that consists of relativistic degenerate matter.

- $n = s = 0$: In this case opacity is dominated by scattering off free electrons (Thomson scattering).

- $n = 1, s = 3.5$: In this case opacity is dominated by the free-free or bound-free absorption.

- $\lambda = 1, \nu \sim 3-6$: This is valid when the pp-chain is responsible for production of nuclear energy.

- $\lambda = 1, \nu \sim 13-23$: This is valid when nuclear energy production is dominated by the CNO-cycle.

The power-law approximations allow us to rescale variables in the stellar equations and reduce them to dimensionless quantities:

$P = P_c\, p$, $T = T_c\, t$, $r = R\, x$, $L_r = L\, \ell$, $M_r = M m$

Here p, t, x, ℓ, m are dimensionless. P_c, T_c are pressure and temperature at the center respectively. R, L, M are determined in terms of P_c, T_c according to the following conditions:

$$\frac{GM^2}{4\pi P_c R^4} = 1 \quad , \quad \frac{M}{4\pi \rho_0 R^3}\frac{T_c^b}{P_c^a} = L \quad , \quad \frac{3\kappa_0 \rho_0^n}{64\pi^2 a c}\frac{P_c^{an}\, ML}{T_c^{4+s+nb} R^4} = 1$$

Focusing on stars with radiative transport as the main mode of energy transport, the stellar equations become:

$$\begin{cases} \dfrac{dp}{dm} = -\dfrac{m}{x^4} \\ \dfrac{dx}{dm} = \dfrac{t^b}{x^2 p^a} \\ \dfrac{dt}{dm} = -\dfrac{p^{an}\,\ell}{x^4 t^{3+s+bn}} \\ \dfrac{d\ell}{dm} = A\, p^{a\lambda}\, t^{\nu - b\lambda} \end{cases} \qquad *$$

Here A is given by:

$$A = \frac{3 K_0 \rho_0^{n} \varepsilon_0}{16\pi G a c} \frac{P_c^{an + a\lambda + 1}}{T_c^{4 + s + nb + \lambda b - \nu}}$$

These four first-order differential equations require four boundary condition for their solutions. Two of these come from definition:

$p(0) = 1$, $t(0) = 1$

And regular behaviour at the origin dictates the other two:

$r(0) = 0$, $l(0) = 0$

Physically, we expect that p and t (hence P and T) vanish at the same value of m (corresponding to the surface of the star). This will be satisfied only for a specific value of A. For that value, we obtain a relation between P_c and T_c:

$$P_c^{a(n+\lambda)+1} \propto T_c^{4+s+b(n+\lambda)-\nu}$$

It is easy to see that the set of equations in * have power-law solutions (as a function of m). The physical quantities R, M, L will have the following form:

$$\boxed{R \propto M^{n_R} \quad , \quad T \propto M^{n_T} \quad , \quad L \propto M^{n_L}}$$

These are called homology relations. For a fully radiative star, which we have considered so far, we find (omitting the straightforward algebra):

$$n_R = \frac{1}{3}\left[1 - \frac{2}{D_r}\left(\frac{b}{a} + \nu - s - 4\right)\right]$$

$$n_T = -\frac{2}{D_r}\left(\frac{1}{a} + \lambda + n\right)$$

$$n_L = 1 + \frac{1}{D_r}\left[2\lambda\left(\frac{b}{a} + \nu - s - 4\right) - 2\nu\left(\frac{1}{a} + \lambda + n\right)\right]$$

Where:

$$D_r = \left(\frac{3}{a} - 4\right)(\nu - s - 4) - \frac{b}{a}(3\lambda + 3n + 4)$$

For a star with convective transport of energy:

$$n_R = \frac{1}{3}\left(1 - \frac{2}{D_c}\right), \quad n_T = \frac{4}{3D_c}, \quad n_L = 1 + \frac{2}{D_c}\left(\lambda + \frac{2}{3}\nu\right)$$

Where:

$$D_c = \left(\frac{3}{a} - 4\right) + \frac{2b}{a}$$

The homology relations allow us to determine the slope of the curve in the $L - T_e$ plane (the H-R diagram):

$$\boxed{T_e \propto \left(\frac{L}{R^2}\right)^{\frac{1}{4}} \propto L^{n_{HR}}, \quad n_{HR} = \frac{1}{4}\left(1 - \frac{2n_R}{n_L}\right)}$$

As it turns out, homologous models do a fair job predicting n_R, n_L, n_{HR} in high mass stars $M > M_\odot$. However, low mass stars seem to not follow any simple homology relations. Hence it is not possible to model them along these lines in any realistic manner.

It is also important to underline the important role of convection and opacity in stellar modeling. The indices in the power-law approximation depend rather sensitively on the nature of opacity within the bulk of the star. Thus accurate stellar modeling requires a good understanding of opacity under different conditions. Regarding convection, the situation is different for high mass and low mass stars. For the former, the convective core does influence the indices to some extent. However, the bulk of the relationship arises from the outer radiative envelope, which affects many of the observed properties like T_e and L. In the low mass stars, convection plays a significant role, and hence accurate stellar modeling is very complicated. Even in those stars, the extreme outer layers are radiative, which plays a crucial role in determining various indices.